# Acoustic eigenvalues of rectangular rooms with arbitrary wall impedances using the interval Newton/generalized bisection method


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Modal analysis of a rectangular room requires evaluation of the eigenvalues of the Helmholtz operator while taking into account the boundary conditions imposed on the walls of the room. When the walls have finite impedances, the acoustic eigenvalue equation becomes complicated and a numerical method that can find all roots within a given interval is required to solve it. In this study, the interval Newton/generalized bisection (IN/GB) method is adopted for solving this problem. For an efficient implementation of this method, bounds are derived for the acoustic eigenvalues and their asymptotic behavior explored. The accuracy of the IN/GB method is verified for a canonical problem by comparing the modal solution with the corresponding finite element solution. Furthermore, reverberation times estimated using the IN/GB method are compared to those calculated using the finite difference method. Through these examples, it is demonstrated that the IN/GB method provides a useful and efficient approach for estimating the acoustic responses of rectangular rooms with finite wall impedances.



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## I. INTRODUCTION

Modal analysis is a classical method for solving problems in room acoustics (see, for example, Refs. 1–5). Using this method, once all the normal modes are known, the acoustic pressure distribution for an arbitrary sound source in a room can be easily computed. Although the modal theory of room acoustics was established and fully formulated over a half century ago,[1] it is still incomplete in the sense that there is no well-developed, general method for finding eigenvalues that correspond to room modes for walls with arbitrary impedances. Only for rooms with perfectly or nearly rigid walls or rooms with the same impedance on each pair of parallel walls are the eigenvalues or their approximations easy to evaluate. Hence, only these cases have typically been considered in the acoustics literature.[1–6] However, the effect of finite wall impedances on quantities of interest, such as the reverberation time, is of general interest and important for real-world proT.dr.esos2t2bedan.esos2t2(rn.esos2t25(ssos2t2aissos2t2neerld)-396.(or)-396.anor)-396.effi-nt

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The remainder of this paper is organized as follows. In Sec. II, we derive the acoustic eigenvalue equation. In Sec. III, we discuss the application of the IN/GB method to the acoustic eigenvalue problem. In particular, we derive limits and approximations of eigenvalues that yield “good” initial guesses for intervals used in this method. These limits and approximations are useful not only for efficient implementation of the IN/GB method, but find more general applications. The subsequent sections describe three numerical experiments. In Sec. IV, we evaluate acoustic eigenvalues using the IN/GB method for a one-dimensional problem. In Sec. V, we evaluate the modal solution for a point source problem and compare our results with a benchmark solution obtained using the finite element method (FEM). In Sec. VI, we estimate room reverberation times using the acoustic eigenvalues, and compare our results with calculations using the finite-difference time-domain (FDTD) method.[13] We make concluding remarks in Sec. VII. In the Appendices, we review interval arithmetic first, and then describe the IN/BG method for single and multiple variable problems, of which the latter is required for solving the acoustic eigenvalue equation.

## II. ACOUSTIC EIGENVALUE EQUATIONS

Normal modes of a rectangular room are obtained by solving the homogeneous Helmholtz (reduced wave) equation. For a room with uniform impedance on each of its walls, the three-dimensional homogeneous Helmholtz problem for the acoustic pressure $p(x,y,z)$ is described as

$$-\nabla^2 p - k^2 p = 0, \quad \text{in } \Omega, \tag{1}$$

$$\nabla p \cdot \mathbf{n}_i = -\frac{i\omega\rho}{Z_i} p = -ik\beta_i p, \quad \text{on } \Gamma_i, \tag{2}$$

where $\Omega$ is the entire space of the room, $\Gamma_i$ is the $i$th wall whose impedance is denoted by $Z_i$, $i$ is the imaginary unit, $k = \omega/c$ is the driving wave number with angular frequency $\omega$, $c$ is the speed of sound, $\mathbf{n}_i$ is the outward normal unit vector on the walls, $\rho$ is the density of the medium inside the room, and $\beta_i = \rho c/Z_i$ are the specific acoustic admittances of the walls. Figure 1 shows the coordinate system used for this problem. The domain is $\Omega = (0,L_x)\times(0,L_y)\times(0,L_z)$.

The solution of this homogeneous Helmholtz problem can be obtained by the separation of variables.[1–5] Exponentials are chosen as the eigenfunctions in this study; for example, the eigenfunction in the $x$ direction is given in the form

$$X(x) = A e^{ik_x x} + B e^{-ik_x x}, \tag{3}$$

where $k_x$ (which is generally a complex value) is the eigenvalue in the $x$ direction, and $A$ and $B$ are complex constants. After applying Eq. (3) to the boundary conditions Eqs. (2) at $x=0$ and $x=L_x$, the $n$th eigenfunction in the $x$ direction is obtained as

$$X_n(x) = (k_{xn} + k\beta_{x1}) e^{ik_{xn}x} + (k_{xn} - k\beta_{x1}) e^{-ik_{xn}x}, \tag{4}$$

where $\beta_{x1}$ is the specific acoustic admittance at $x=0$. $k_{xn}$ is the $n$th root of $k_x$ for the following acoustic eigenvalue equation:

[illegible]

$$
\begin{aligned}
e_1(\,_R, \,_I) \\
&\equiv (\,_R - \,_{1R})(\,_R - \,_{2R}) - (\,_I - \,_{1I})(\,_I - \,_{2I}) \\
&\quad - \,^{-2\,_I L}\{\cos(2\,_R L)[(\,_R + \,_{1R})(\,_R + \,_{2R}) \\
&\quad - (\,_I + \,_{1I})(\,_I + \,_{2I})] - \sin(2\,_R L)[(\,_R + \,_{1R}) \\
&\quad \times (\,_I + \,_{2I}) + (\,_I + \,_{1I}
\end{aligned}
$$

Substituting the perturbation expansion of in Eq. (10) into Eqs. (13) and (14) and focusing on the terms of $O(\ ^0)$ shows that

$$(\ _0^2 - 1)[\ ^{-2\ _I L}\cos(2\ _R L) - 1] + 2\ _0\ ^{-2\ _I L}\sin(2\ _R L) = 0, \tag{15}$$

$$2\ _0[\ ^{-2\ _I L}\cos(2\ _R L) - 1] - (\ _0^2 - 1)\ ^{-2\ _I L}\sin(2\ _R L) = 0. \tag{16}$$

The elimination of $\ _0$

$$= \int_0^{L} {}^{2}\, \mathrm{d} \qquad (25)$$

$$= 2L\,[\;{}^{2} - (\;\;{}_{1})^{2}] - \frac{}{2}[(\;\; + \;\;{}_{1})^{2}(\;{}^{2}\,{}^{L} - 1)$$
$$- (\;\; - \;\;{}_{1})^{2}(\;{}^{-2}\,{}^{L} - 1)]. \qquad (26)$$

Note that in Eq. (24), is multiplied by , and not by the complex conjugate of . This orthogonality can be proven by explicitly evaluating the integral in Eq. (24) and using the fact that satisfy the corresponding acoustic eigenvalue equation, Eq. (5). The eigenfunction is normalized as

$$(\;) = \frac{(\;)}{\sqrt{\;\;}}. \qquad (27)$$

The modal solution is represented in terms of the orthonormal eigenfunctions as

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$$(\;,\;) = \sum_{=1}$$

evaluate tradeoff between the accuracy of the modal solution and the number of terms included in the summation, and the IN/GB method can be used to find acoustic eigenvalues with whatever precision is desirable or necessary.

## VI. NUMERICAL E PERIMENT 3: ROOM REVERBERATION TIMES

As another example application of finding acoustic eigenvalues using the IN/GB approach, room reverberation times were calculated for three-dimensional rectangular rooms and then compared with the results obtained from the finite-difference time-domain (FDTD) method reported by Yasuda . in Ref. 13. The width and the depth of the room were $L = 24$ m, $L = 12$ m, respectively, and the height was either $L = 3$ m or $L = 6$ m. An absorber with absorption coefficient $= 0.5$ was installed either only at $= 0$ or both at $= 0$ and $= L$. All other walls were assumed to have $= 0.05$. The corresponding specific acoustic impedances were all given as real values, i.e.,

$$\frac{}{} = \frac{1 + \sqrt{1 - }}{1 - \sqrt{1 - }}. \qquad (37)$$

For these conditions, the reverberation times in 1/3 octave bands were calculated using the eigenvalues obtained by the IN/GB method.

In order to estimate the reverberation times, the collective modal decay curves were first obtained by

$$\langle \, ^2\rangle( \quad ) = 10 \log\left(\frac{\sum \, ^{-2 \quad / \quad I}}{\sum \, 1/ \quad I}\right), \quad \text{in dB}, \qquad (38)$$

where is time, is a trio of , , and (the mode numbers in the , , and directions, respectively), $_I$ is the damping constant (the imaginary part of the eigenvalue ), and the summation is over all eigenvalues whose eigenfrequencies (real parts of ) are within the band .[7,18] Although this decay curve is not exact, it roughly characterizes the energy decay with time in a given frequency band. Figure 6 shows the decay curves obtained from Eq. (38) in 1/3 octave bands for all height and absorber configuration conditions mentioned above. The driving wave numbers were set such that they correspond to the center frequencies of the 1/3 octave bands.

The reverberation times $_{60}$ were obtained from the modal decay curve, Fig. 6, and are plotted in Fig. 7 along with the reverberation times computed in Ref. 13, which used the FDTD approach to solve the same problem. A comditions for each51.1(each)m00rod

$$ {}^{(,)} = (\;^{(,)}), \tag{C4} $$

$$ N(\;^{(,)}, \;^{(,)}) = \;^{(,)} - \frac{e(\;^{(,)})}{e'(\;^{(,)})}, \tag{C5} $$

$$ {}^{(,+1)} = \;^{(,)} \cap N(\;^{(,)}, \;^{(,)}), \tag{C6} $$

where ( ) is the midpoint of , calculated for $=[\;,$

[11] R. B. Kearfott, *Rigorous Global Search: Continuous Problems* (Kluwer Academic, Dordrecht, 1996).

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