Basic Examination Sample
Measure and Integration

Solve three of the following problems.

1. State and prove Egoroff's theorem.

2. Let $E \subset$ be a Lebesgue measurable set with $\mathcal{L}^1(E) > 0$. Prove that for every $0 < t < \mathcal{L}^1(E)$ there exists a Lebesgue measurable subset $F \subset E$ such that $\mathcal{L}^1(F) = t$.

3. Consider the function

$$F(y) = \int_0^1 \frac{e^{yx}}{1 + x^2}\, dx; \quad y \geq 0.$$

   (a) Prove that F is continuous.

   (b) Prove that F is differentiable for $y > 0$.

   (c) Prove that F' is differentiable for $y > 0$.

   (d) Prove that $F''(y) + F(y) = \frac{1}{y}$ for all $y > 0$.

4. Let $f : \to$ be a differentiable function. Assume that there exists $M \geq 0$ such that

$$|f'(x)| \leq M$$

   for all $x \in [a; b]$ for some $a < b$.

   (a) Prove that f' is Borel measurable.

   (b) Prove that

$$\lim_{n \to \infty} \quad f \quad f$$