Algebra basic exam, September 2023

180 minutes
Each of the five questions is worth the same.

1. (a) Prove that $\mathbb{Z}[\sqrt{-2}]$ is a principal ideal domain.
   (b) Factor 11 into irreducible elements in the ring $\mathbb{Z}[\sqrt{-2}]$. Explain why the factors are irreducible in this ring.

2. True or false: Suppose E and F are fields and $\varphi : E^\times \to F^\times$ is an injective group homomorphism. Then φ extends to a ring homomorphism $E \to F$. Justify.

3. Give two distinct examples of pairs (R, M), where R is a commutative ring with 1 and M is an R-module that is torsion-free, but not free. The rings R in these examples must be different (non-isomorphic). Explain why these examples have the required properties.

4. Do one (and only one) of the following problems.

   (a) State and prove Buchberger's criterion. You may assume the division algorithm for several polynomials as already defined.
   (b) Define the term "Sylow p-subgroup" and prove that, if the order of a finite group G is divisible by p, then G has a Sylow p-subgroup.

   Clearly indicate which problem you chose.

5. Let $G = F(x, y)$ be the free group on symbols x and y. Define a subgroup H of G such that G/H is isomorphic to S_5. Explain why $G/H \cong S_5$ in your example.