ALGEBRA BASIC EXAM: SAMPLE

Attempt four of the following six questions. All questions carry equal weight. All rings are assumed to be commutative rings with 1.

(1) State and prove the Sylow theorems.

(2) Prove that if R is a PID then every fg R-module is a direct sum of cyclic R-modules. Use this to prove that every complex matrix has a Jordan canonical form.

(3) State the Fundamental Theorem of Galois theory. Let F be the unique subfield of $\mathbb{C}$ which is a splitting field for $x^4 - 2$ over $\mathbb{Q}$. Find $[F : \mathbb{Q}]$. Describe the Galois group of F over $\mathbb{Q}$. Find all the fields which are intermediate between $\mathbb{Q}$ and F.

(4) Show that every PID is a UFD. Show that $\mathbb{Z}[\sqrt{10}]$ is not a UFD. Find all the prime ideals P of $\mathbb{Z}[\sqrt{10}]$ such that $(3) \subseteq P$.

(5) Show that every proper ideal of a ring R is contained in a maximal ideal, and that every maximal ideal is prime. Prove that the following are equivalent for an element r of a ring R:
   (a) $1 + rs$ is a unit for all $s \in R$.
   (b) r is in every maximal ideal of R.

(6) Let p be an odd prime and let $\zeta = e^{2\pi i/p}$, $\alpha = \zeta + \zeta^{-1}$. Show that $\mathbb{Q}(\alpha)$ is a Galois extension of $\mathbb{Q}$ and find its degree over $\mathbb{Q}$. For $p = 7$ find the minimal polynomial of α over $\mathbb{Q}$.