This test is **closed book**: No books, notes, or access to any other relevant materials (including Internet consultation) are permitted.

You have 3 hours. The exam has a total of 5 questions and 100 points (20 each).

You may use without proof *standard* results from the syllabus (not from homework) which are independent of the question asked, unless explicitly instructed otherwise. You must, however, clearly **state** the result you are using.

Problem 1 : Assuming that $u : \mathbb{R}^3 \to \mathbb{R}$ is harmonic, prove that the vector field

$$v(x) = (u + 2x \cdot Du)Du - x|Du|^2, \qquad x \in \mathbb{R}^3,$$

has zero divergence, then use this to prove that for every $r > 0$,

$$\int_{B(0,r)} |Du|^2\,dx \le r \int_{\partial B(0,r)} |Du|^2\,dS.$$

Finally, deduce from this that the function

$$r \mapsto \frac{1}{r}\int_{B(0,r)} |Du|^2\,dx, \qquad r > 0,$$

is non-decreasing. (Notation: Du is the gradient of u, and $B(0,r) = \{x \in \mathbb{R}^n : |x| \le r\}$.)

Problem 2 : Let U be a bounded open set in $\mathbb{R}^n$ with smooth boundary, and let $u(x,t)$ be a smooth solution of

$$\begin{aligned} u_t - \Delta u + c(x,t)u &= 0, & x \in U,\ t > 0,\\ u(x,t) &= 0, & x \in \partial U,\ t > 0,\\ u(x,t) &= g(x), & x \in U,\ t = 0. \end{aligned}$$

Assume $c(x,t)$;r> ;

Problem 4 : Suppose $u\colon \mathbb{R} \times [0,T] \to \mathbb{R}$ is a smooth solution of $u_t + uu_x = 0$ that is periodic in x with period L, i.e., $u(x+L,t) = u(x,t)$. Show that

$$\max_x u(x,0) - \min_x u(x,0) \le \frac{L}{T}.$$

Problem 5 : Consider a scalar conservation law

$$u_t + f_\varepsilon(u)_x = 0, \qquad x \in \mathbb{R},\ t > 0,$$

with convex flux function given by $f_\varepsilon(u) = \sqrt{u^2 + \varepsilon}$ where $\varepsilon > 0$, and initial data

$$u(x,0) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

(i) Find a discontinuous weak solution of this problem, and determine whether the Lax entropy condition holds at points of discontinuity.

(ii) Show that a continuous solution $u^\varepsilon(x,t)$ exists in the form of a centered rarefaction wave for $t > 0$. Find $u^0(x,t) = \lim_{\varepsilon \to 0} u^\varepsilon(x,t)$ and show that u^0 is a weak solution of

$$u_t + |u|_x = 0, \qquad x \in \mathbb{R},\ t > 0.$$