Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

BASIC EXAMINATION: FUNCTIONAL ANALYSIS

September 2, 2014, Wean Hall 7201, 4:30pm-6:30pm

1. (i) State the Closed Graph Theorem.

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces.

(ii) Let $T : X \to Y$ be a linear and continuous operator such that Range(T) is closed. Prove that there exists $C > 0$ such that for every $y \in T(X)$ there exists $x \in X$ such that

$$y = T(x) \quad \text{and} \quad \|x\|_X \le C\|y\|_Y .$$

(iii) Let $T : X \to Y$ be a linear operator such that for every sequence $\{x_n\} \subset X$

$$\|x_n\|_X \to 0 \Rightarrow T(x_n) \rightharpoonup 0 \quad \text{in} \quad (Y, Y').$$

Prove that T is continuous, i.e., $T \in L(X;Y)$.

2. Prove that if $(X, \|\cdot\|)$ is a normed space over $\mathbb{R}$ such that X' is separable, then X is also separable.

3. (i) State and prove the Banach-Steinhaus Theorem for normed spaces.

(ii) Let $(X, \|\cdot\|_X)$ be a Banach space over $\mathbb{R}$ and let $L_n, L \in X', n \in \mathbb{N}$, be such that $L_n \overset{*}{\rightharpoonup} L$. Prove that

$$\|L\|_{X'} \le \liminf_{n \to \infty} \|L_n\|_{X'} < +\infty .$$

4. (i) Give the definition of a compact operator between two normed spaces.

(ii) Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces, and let $T : X \to Y$ be a linear compact operator. Prove that T^* is also compact.

5. Let $(X, ($