Department of Mathematical Sciences
Carnegie Mellon University

Basic Examination
Introduction To Functional Analysis
September 2021

Time allowed: 180 minutes.

This test is closed book: no notes or other aids are permitted. You may use without proof standard results from the syllabus which are independent of the question asked, unless explicitly instructed otherwise. You must, however, clearly state the result you are using.

1. Let $(X, \|\cdot\|)$ be a normed space and let $T : X \to X$ be a linear operator.
   (a) Define what it means for T to be compact.
   (b) Let $T : X \to X$ be a linear compact operator. Prove that either the homogeneous equation
   $$x - T(x) = 0$$
   has a nontrivial solution $x \in X \setminus \{0\}$ or for every $y \in X$ the equation
   $$x - T(x) = y$$
   has a unique solution $x \in X$.[1]

2. Let $(X, \|\cdot\|)$ be an infinite dimensional Banach space and $T : X \to \mathbb{R}$ a linear discontinuous function. Define
   $$\|x\|_T := \|x\| + |T(x)|, \quad x \in X.$$
   Prove that $\|\cdot\|_T$ is a norm and that $(X, \|\cdot\|_T)$ is not a Banach space.

3. Let $(X, \|\cdot\|)$ be a Banach space, $Y \subset X$ a subspace, and $L_0 \in X'$. Prove that there exists $L_1 \in Y^\perp$ such that
   $$\inf_{L \in Y^\perp} \|L - L_0\|_{X'} = \|L_1 - L_0\|_{X'},$$
   where $Y^\perp = \{L \in X' : L(x) = 0 \text{ for all } x \in Y\}$.

4. Let $(X, \|\cdot\|)$ be a reflexive Banach space and $Y \subset X$ be a closed subspace. Prove that $(Y, \|\cdot\|)$ is reflexive.

5. Let $1 < p < \infty$ and consider the space $L^p([0,1])$, where the underlying measure is the Lebesgue measure. Consider the sequence of functions
   $$f_n(x) = \sin(2\pi n x), \quad x \in [0,1].$$
   Prove that $\{f_n\}_n$ converges weakly in $L^p([0,1])$, but not strongly in $L^p([0,1])$.

[1] Don't just quote the Fredholm alternative theorem. I am asking you to prove the Fredholm alternative