Department of Mathematical Sciences
Carnegie Mellon University

Basic Examination
Introduction To Functional Analysis
September 2023

Time allowed: 180 minutes.

This test is closed book: no notes or other aids are permitted. You may use without proof standard results from the syllabus which are independent of the question asked, unless explicitly instructed otherwise. You must, however, clearly state the result you are using.

1. Let X be a Hilbert space and let $L \in X'$. Consider the function

$$f(x) := \|x\|^2 - L(x), \quad x \in X.$$

   (a) Prove that there exists $y_0 \in X$ such that

$$f(x) = \|x - y_0\|^2 - \|y_0\|^2.$$

   (b) Let $C \subset X$ be a nonempty closed and convex set. Prove that that exists a unique $x_0 \in C$ such that

$$f(x_0) = \min_{x \in C} f(x).$$

2. Let Y be a Banach space.

   (a) Let $\{y_n\}_n$ be a sequence in Y converging weakly. Prove that $\{y_n\}_n$ is bounded.

   (b) Let $Y = L^2([0,1])$ and let $\{f_n\}_n$ be a sequence converging weakly to f. Let

$$F_n(x) := \int_0^x f_n(t)\,dt, \quad F(x) := \int_0^x f(t)\,dt.$$

   Prove that F_n and F are continuous and that $F_n \to F$ uniformly in $[0,1]$.

3. Let X be a Banach space. Prove that if X' if reflexive, then X is reflexive.

4. Let $X = C([0,1])$ be the Banach space of all continuous functions $f : [0,1] \to \mathbb{C}$, with the sup norm. Define $T \in \mathcal{L}(X)$ by

$$T(f)(x) := f(x) + \int_0^x f(y)\,dy.$$

   Find the spectrum $\sigma(T)$.