Department of Mathematical Sciences
Carnegie Mellon University

Functional Analysis

Sample Exam

Do any 4 of the following 6 problems. All problems carry equal weight.

1. (a) Let X be a Banach space, Y be a normed linear space, and $\{T_n\}_{n=1}^{\infty}$ be a sequence of bounded linear mappings from X to Y satisfying

$$\forall\, x \in X, \quad \sup\{\|T_n x\| : n \in N\} < \infty.$$

Prove that

$$\sup\{\|T_n\| : n \in N\} < \infty.$$

(Do not simply quote the Banach-Steinhaus Theorem (aka the Principle of Uniform Boundedness). You are being asked to prove that theorem.)

(b) Give an example of a normed linear space X, a Banach space Y, and a sequence $\{T_n\}_{n=1}^{\infty}$ of bounded linear mappings from X to Y satisfying

$$\forall\, x \in X, \quad \sup\{\|T_n x\| : n \in N\} < \infty,$$

but

$$\sup\{\|T_n\| : n \in N\} = \infty.$$

2. (a) Let X be a reflexive Banach space and let $\{K_n\}_{n=1}^{\infty}$ be a sequence of bounded subsets of X

4. (a) State the Open Mapping Theorem and the Closed Graph Theorem.

   (b) Use the Open Mapping Theorem to Prove the Closed Graph Theorem.

   (c) Let X, Y, Z be Banach spaces and $U : X \to Y$, $V : Y \to Z$ be linear mappings and define $T : X \to Z$ by $Tx = VUx$ for all $x \in X$. Assume that T is continuous and that V is continuous and injective. Prove that U is continuous.

5. (a) Let X be a Banach space and $T : X \to X$ be a linear mapping such that $T^2 = T$. Show that T is continuous if and only if the null space and range of T both are closed.

   (b) Let X, Y be Banach spaces and put

$$\mathcal{O} = \{T \in \mathcal{L}(X;Y) : T^*[Y^*] = X^*\}.$$

Show that $\mathcal{O}$ is an open subset of $\mathcal{L}(X;Y)$ (equipped with the operator norm). Here $\mathcal{L}(X;Y)$ is the set of all bounded linear mappings from X to Y, X^* and Y^* are the (topological) duals of X and Y, and $T^* \in \mathcal{L}(Y^*;X^*)$