Basic Examination Sample
General Topology

Do four of the five problems.

1. Consider the topology on $\mathbb{R}$ for which $\mathcal{A} = \{[a,b) : a,b \in \mathbb{R}\}$ is a subbasis.

   (a) Prove that for all $x \in \mathbb{R}$ there exists a countable local basis.

   (b) Show that the space does not have a countable basis of topology.

2. Show that every compact Hausdorff topological space is normal.

3. Let $f : [a,b] \to \mathbb{R}$ and let

   $$\operatorname{gr} f := \{(x, f(x)) : x \in [a,b]\}$$

   be the graph of f.

   Prove that the following two conditions are equivalent:

   (i) f is continuous.

   (ii) $\operatorname{gr} f$ is compact.

4. Let $f_n : [0,1] \to [0,1]$ be a sequence of functions such that for all n and all $x,y \in [0,1]$ such that $|x - y| > 1/n$

   $$|f_n(x) - f_n(y)| \leq \frac{1}{n}|x - y|.$$

   Show that f_n has a uniformly convergent subsequence.

5. Let (X,d) be a complete metric space and $S : X \to X$ such that $S^2 := S \circ S$ is a strict contraction. That is, there exists $\lambda \in [0,1)$ such that

   $$d(S^2(x), S^2(y)) \leq \lambda d(x,y) \quad \text{for all } x,y \in X.$$

   Show that the mapping S has exactly one fixed point.