Department of Mathematical Sciences
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Basic Examination
Measure and Integration
January 2012

Time allowed: 120 minutes.
Do four of the five problems. Indicate on the first page which problems you have chosen to be graded. All problems carry the same weight.

1. Let X be a nonempty set, let $\mathcal{M}$ be an algebra on X, and let $f : X \to [0, \infty]$ be a measurable function. Prove that there exists a sequence $\{s_n\}$ of simple functions such that
$$0 \le s_1(x) \le s_2(x) \le \dots \le s_n(x) \to f(x)$$
for every $x \in X$ and that the convergence is uniform on any set on which f is bounded from above.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $1 \le p < \infty$. Prove that $L^p(X)$ is a complete metric space.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function, integrable on compact sets, and for every $\varepsilon > 0$, let
$$f_\varepsilon(x) := \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f(t)\, dt.$$
   (a) Prove that
$$\int_R |f_\varepsilon(x)|\, dx \le \int_R |f(x)|\, dx.$$
   (b) Prove that if $f \in C(\mathbb{R})$, then $f_\varepsilon(x) \to f(x)$ as $\varepsilon \to 0^+$ for every $x \in \mathbb{R}$.
   (c) Prove that if f is integrable, then $f_\varepsilon(x) \to f(x)$ as $\varepsilon \to 0^+$ for $\mathcal{L}^1$ a.e. $x \in \mathbb{R}$.

4. Given the function $f : [0,1] \times [0,1] \to \mathbb{R}$ defined by
$$f(x, y) = \begin{cases} \dfrac{1}{\left(x - \frac{1}{2}\right)^3} & \text{if } 0 < y < \left|x - \frac{1}{2}\right|, \\ 0 & \text{otherwise,} \end{cases}$$
determine if the integrals
$$\int_0^1 \left(\int_0^1 f(x, y)\, dx \right) dy; \quad \int_0^1 \left(\int_0^1 f(x, y)\, dy \right) dx; \quad \int_{[0,1]^2} \quad \text{if } 0$$

(a) Prove that the set function $\sum_{=1}^{1}$: M ! [0; 1], de…ned by

$\sum_{=1}^{1}$ (*E*) := sup $\sum_{=1}^{1}$ (*E*) : *E* 2 M; *E* pairwise disjoint, $\bigcup_{=1}^{1}$ *E* = *E* ; *E* 2 M;

is a measure.

(b) Prove that if : M ! [0; 1] is a measure such that for every *n*, then $\sum_{=1}^{1}$. E (*E*) : *E1* n

(c) Prove that if is …nite and each : M n