Basic Examination: Measure and Integration— August 2017

This test is closed book: no notes or other aids are permitted.

You have 3 hours. The exam has a total of 5 questions and 100 points (20 each).

You may use without proof standard results from the syllabus which are independent of the question asked, unless explicitly instructed otherwise. You must, however, clearly state the result you are using.

1. State and prove the Radon-Nikodym theorem.
2. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \to [0, \infty]$ be a measurable function. Consider the function $\varphi_f : (0, \infty) \to [0, \mu(X)]$, defined by
$$\varphi_f(s) = \mu(\{x \in X : f(x) > s\}), \quad s > 0.$$
    (a) Prove that φ_f is decreasing and continuous from the right.
    (b) Assume that $\varphi_f(s) < \infty$ for every $s > 0$. Prove that φ_f is continuous at $s > 0$ if and only if $\mu(\{x \in X : f(x) = s\}) = 0$.
    (c) Prove that
$$\int_X f \, d\mu = \int_0^\infty \varphi_f(s) \, ds.$$
3. Let $f : [a, b] \to \mathbb{R}$ be a Lebesgue integrable function.
    (a) Prove that if
$$\int_E f(x) \, dx = 0$$
    for every Lebesgue measurable set $E \subseteq [a, b]$, then $f(x) = 0$ for $\mathcal{L}^1$ a.e. $x \in [a, b]$.
    (b) Prove that if
$$\int_a^b f(x) g(x) \, dx = 0$$
    for all $g \in C([a, b])$, then $f(x) = 0$ for $\mathcal{L}^1$ a.e. $x \in [a, b]$.
    (c) Prove that if
$$\int_a^b f(x) g(x) \, dx = 0$$
    for all $g \in C([a, b])$ with $\int_a^b g(x) \, dx = 0$, then there exists a constant $c \in \mathbb{R}$ such that $f(x) = c$ for $\mathcal{L}^1$ a.e. $x \in [a, b]$.
    (d) Prove that if
$$\int_a^b f(x) h'(x) \, dx = 0$$
    for all $h \in C^1([a, b])$ with $h(a) = h(b) = 0$, then there exists a constant $c \in \mathbb{R}$ such that $f(x) = c$ for $\mathcal{L}^1$ a.e. $x \in [a, b]$.

4. Let ' ! ; 1 be a convex function such that ' and $\frac{'(t)}{t}$ t ! 1 1 . For every Lebesgue measurable function **f** ; ! de...ne

$$\text{k}\boldsymbol{f}\text{k} \quad \left\{ s > \int_0^1 ' \ f \ x = s \ dx \right\};$$

where we set ; 1 .

(a) Given a Lebesgue measurable function **f** ; ! , prove that k**f**k if and only if **f x** for L^1-a.e. **x** 2 ; .

(b) Let **L** ; be given by all (equivalence classes of) Lebesgue measurable functions **f** ; ! such that k**f**k **<** 1 . Prove that k**f**k **<** 1 if and only if $\int_0^1$ ' **f x dx** and deduce that k k is a norm in **L** ; .

(c) Prove that **L** ; **L**^{1} ; and deduce that **L** ; is a Banach spL