Basic Examination: Measure and Integration— January 2018

This test is closed book: no notes or other aids are permitted.

You have 3 hours.

You may use without proof standard results from the syllabus which are independent of the question asked, unless explicitly instructed otherwise. You must, however, clearly state the result you are using.

1. (15 points) State and prove the Lebesgue monotone convergence theorem.

2. Let $\mu : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ be a measure absolutely continuous with respect to the Lebesgue measure $\mathcal{L}$.

   (a) (5 points) Prove or give a counter example. If $E_n \subset \mathbb{R}$ are Borel sets with $\mathcal{L}(E_n) \to 0$, then $\mu(E_n) \to 0$ as $n \to \infty$.

   (b) (10 points) Prove or give a counter example. If μ is a finite measure and $E_n \subset \mathbb{R}$ are Borel sets with $\mathcal{L}(E_n) \to 0$, then $\mu(E_n) \to 0$ as $n \to \infty$.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space. Consider the sequence of functions

$$f_n = a_n \chi_{E_n},$$

   where the sets $E_n \in \mathcal{M}$ are such that $\mu(E_n) > 0$ for all $n \in \mathbb{N}$ and $a_n > 0$ are such that either $a_n \to 0$ as $n \to \infty$ or $a_n \geq c$ for all $n \in \mathbb{N}$ and for some $c > 0$. Prove that

   (a) (4 points) $f_n \to 0$ in measure if and only if $a_n \to 0$ or $\mu(E_n) \to 0$ as $n \to \infty$,

   (b) (3 points) $f_n \to 0$ in $L^p(X)$, $1 \leq p < \infty$, if and only if $a_n^p \mu(E_n) \to 0$ as $n \to \infty$,

   (c) (6 points) $f_n \to 0$ pointwise almost everywhere if and only if $a_n \to 0$ as $n \to \infty$ or the set

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

   has measure zero,

   (d) (7 points) $f_n \to 0$ almost uniformly if and only if $a_n \to 0$ as $n \to \infty$ or

$$\mu\left(\bigcup_{k=n}^{\infty} E_k\right) \to 0$$

   as $n \to \infty$.

4. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set.

(a) (8 points) Consider the space $L^1(E) + L^\infty(E)$ of all Lebesgue measurable functions $f : E \to \mathbb{R}$ such that f can be written as $f = g + h$ for some $g \in L^1(E)$ and $h \in L^\infty(E)$. Given $f \in L^1(E) + L^\infty(E)$, define

$$\|f\| := \inf\{\|g\|_1 + \|h\|_\infty\},$$

where the infimum is taken over all possible decompositions $f = g + h$, where $g \in L^1(E)$ and $h \in L^\infty(E)$. Prove that $\|\cdot\|$ is a norm in $L^1(E) + L^\infty(E)$.

(b) (8 points) Prove that $L^1(E) + L^\infty(E)$ is a Banach space.

(c) (4 points) Let $1 < p < \infty$. Prove that $L^p(E) \subset L^1(E) + L^\infty(E)$.

5. Given the function

$$f(x) = \int_0^\infty e^{-xy^2}\,dy, \quad x \in (0, \infty).$$

(a) (7 points) Use the Lebesgue dominated convergence theorem to prove that f is continuous (without using part (b)).

(b) (13 points) Use the Lebesgue dominated convergence theorem to prove that f is differentiable.