Basic Examination: Measure and Integration January 2020

> This test is **closed book**: no notes or other aids are permitted.
> You have 3 hours. The exam has a total of 5 questions and 100 points (20 each).
> You may use without proof *standard* results from the syllabus which are independent of the question asked, unless explicitly instructed otherwise. You must, however, **clearly** state the result you are using.

Below, if not stated explicitly, $(X, \mathcal{F}, \mu)$ is a measure space, $L_p = L_p(X, \mathcal{F}, \mu)$ is the standard L_p space ($p \in [1, \infty]$) and m is Lebesgue measure.

1. Suppose $I = [0, 1]$, $f_n : I \to \mathbb{R}$ is Lebesgue measurable for all $n \in \mathbb{N}$, and

$$\int_{[0,1]} |f_n|^2 \, dm \le 5 \quad \text{for all } n \in \mathbb{N}.$$

Suppose moreover that $f_n(x) \to 0$ as $n \to \infty$, for every $x \in [0, 1]$.

(a) Does it necessarily follow that $\lim_{n \to \infty} \int_I |f_n|^2 \, dm = 0$?

(b) Does it necessarily follow that $\lim_{n \to \infty} \int_I |f_n| \, dm = 0$?

In each case, prove the implication or find a counterexample.

2. Let $f_n \in L_1$ for all $n \in \mathbb{N}$. Which is always larger,

$$\left(\sum_{n=1}^{\infty} \left(\int f_n \, d\mu \right)^2 \right)^{1/2} \quad \text{or} \quad \int \left(\sum_{n=1}^{\infty} |f_n|^2 \right)^{1/2} d\mu \, ?$$

Prove your answer.

3. Suppose $\mu(X) < \infty$. Let $f : X \to \mathbb{R}$ be $\mathcal{F}$-measurable, and define

$$g(x, y) = f(x) - f(y), \qquad x, y \in X.$$

If g is $\mu \times \mu$-integrable on $X \times X$, is f necessarily μ-integrable? Prove or find a counterexample.

4. Let μ, ν, and ν_n be finite measures on $(X, \mathcal{F})$, and suppose that $\nu_n \ll \mu$ for all $n \in \mathbb{N}$. Is the following true or false?

If