Department of Mathematical Sciences
Carnegie Mellon University

Basic Examination
Probability
Fall 2018

Time allowed: 180 minutes. Justify your answers.

1. Recite precisely the following definitions/facts/theorems/lemmas:
   (a) Dynkin's lemma on π-systems.
   (b) First and second Borel-Cantelli lemmas.
   (c) Kolmogorov's three-series theorem on convergence of sums of IRVs.
   (d) Let (X_n) be a non-negative submartingale. Will it converge (i) almost surely, (ii) in probability, (iii) in L_1, (iv) weakly to some random variable X_∞? If needed, formulate additional (as sharp as possible) conditions on (X_n) under which these convergences take place.

2. Let $X_1, \dots, X_N$ be IID Gaussian RVs with mean 0 and variance 1 and denote $S_N = \sum_{1 \le n \le N} X_n$ and $Y = I(S_N > 0)$. Compute the function $f = f(x)$ such that

$$f(Y) = \mathbb{E}(X_1 | Y).$$

3. Let (X_n) be IID bounded (non-constant) RVs and suppose that a sequence of real numbers (a_n) is chosen so that the series $\sum_n a_n X_n$ converges almost surely.
   (a) Will the series $\sum_n a_n X_n$ converge in L_1?
   (b) Is it true that $\sum_n |a_n| < \infty$?.

4. Let (X_n) be a martingale bounded in L_2 and $(\mathcal{G}_n)$ be the filtration generated by the absolute values of (X_n):

$$\mathcal{G}_n = \sigma(|X_1|, |X_2|, \dots, |X_n|); \quad n \ge 1.$$

   Will the sequence $Y_n = \mathbb{E}(X_n | \mathcal{G}_n)$, $n \ge 1$, converge (a) a.s.? (b) in L_2?

5. Let (X_n) be IID RVs with uniform distribution on $(-1;1)$. Denote

$$S_n = X_1 + \cdots + X_n; \quad S_0 = 0.$$

Show that there is a constant $a > 0$ such that

$$\limsup_n \mathrm{P}\left(\max_{k \le n} S_k \ge c\sqrt{n}\right) \le e^{-ac^2}; \quad \forall c > 0.$$

Write your best possible estimate for a.

6. Let (X_n) be IID RVs with the distribution function

$$\mathrm{P}(X_1 \le x) = \frac{1}{\pi}\int_{-\infty}^{x} \frac{dy}{1+y^2}; \quad x \in \mathbb{R}.$$

There are a constant $p > 0$ and a random variable $Y \ne 0$ such that the sequence

$$Y_n \triangleq \frac{1}{n^p}\sum_{1 \le k \le n} X_k; \quad n \ge 1;$$

converges weakly to Y. Compute p and the distribution function of Y.

7. Let (X_n) be a symmetric random walk on integers starting at 0. For an integer $a > 0$ define the hitting time

$$\tau = \inf\{n \ge 0 : X_n = a\}.$$

Compute the Laplace transform $L(\lambda) = \mathrm{E}\left(e^{-\lambda\tau}\right)$, $\lambda \ge 0$ and the mean $\mathrm{E}(\tau)$.

8. Let (X_n) be IID non-negative RVs, each having density $\phi = \phi(x)$ with respect to the Lebesgue measure and expectation

$$\mathrm{E}(X_1) = \int_0^{\infty} x\,\phi(x)dx = 1.$$

Let $Y_0 = 1$ and $Y_n = \prod_{k=1}^{n} X_k$, $n \ge 1$.

   (a) Obtain a (deterministic) integral equation, which solution yields

$$a(\lambda) = \mathrm{P}\left(\max_{n \ge 0} Y_n \ge 2\lambda\right).$$

   (b) Write your best estimate for $a^* = \sup_\lambda \lambda\, a(\lambda)$.