SET THEORY BASIC EXAM: JANUARY 2017

Attempt four of the following six questions. All questions carry equal weight.

(1) Define the terms *cardinal*, *singular cardinal*, *regular cardinal*, *strong limit cardinal*. Prove that if κ is a singular strong limit cardinal then $2^\kappa = \kappa^{\mathrm{cf}(\kappa)}$. Prove that for any infinite cardinal κ, $\mathrm{cf}(2^\kappa) > \kappa$.

(2) State the Condensation Lemma for L, and give a brief outline of the proof.

Assuming that $V = L$, prove that:

(a) If X is countable with $X \subseteq L_{\omega_1}$, then $X = L_\alpha$ for some countable α.

(b) There is a countable X with $X \subseteq L_{\omega_2}$ which is not of the form L_α for any ordinal α.

(3) Define the terms *Aronszajn tree* and *Souslin tree*.

Assume that T is an ω_1-tree and there exists a function $f : T \to \omega$ such that for all s and t in T, $s <_T t \implies f(s) \neq f(t)$. Prove that:

(a) T is Aronszajn.

(b) T is not Souslin.

(4) Let S

(6) State and prove the Reflection Theorem. Define the class HOD, and outline a proof that HOD is a transitive class model of ZFC set theory. Prove that $\aleph_\omega^V$ is a singular cardinal in HOD.