SET THEORY BASIC EXAM: SAMPLE

JC

Attempt four of the following six questions. All questions carry equal weight.

(1) Let θ be regular. Prove that if κ is a cardinal less than θ and $\kappa \subseteq M \prec H(\theta)$ then $M \cap \kappa^+ \in \kappa^+$. Prove that if $\kappa \leq \lambda$ then $H(\kappa)$ is a Σ_1-elementary substructure of $H(\lambda)$.

Prove that the following are equivalent properties for a set $S \subseteq \omega_1$:

(a) S is stationary.

(b) There is a countable $M \prec H(\omega_2)$ such that $S \in M$ and $M \cap \omega_1 \in S$.

(2) Define the classes HOD and L. Explain carefully why $HOD^L = L^{HOD} = L$. You may use any of the basic theorems about L and HOD, as long as you quote them correctly and explain why they can be applied.

(3) Define the concepts *ω_1-tree*, *Aronszajn tree*, *Souslin tree*, and *special Aronszajn tree*. Prove that Souslin trees are not special.

Let S and T be ω_1-trees. Define $S \otimes T$ to be the set of pairs $(s, t) \in S \times T$ with $ht(s) = ht(t)$, ordered in the natural way.

(a) Show that if S is any ω_1-tree then $S \otimes S$ is not a Souslin tree.

(b) Use $\diamondsuit$ to build Souslin trees S and T such that $S \otimes T$ is Souslin.

(4) Let κ be uncountable and regular. Define the concepts of *club subset of κ* and *stationary subset of κ*, and explain how to extend them to subsets of ordinals of uncountable cofinality. State Fodor's lemma.

Define a relation $<$ on stationary subsets of κ: $S < T$ iff there is a club set $C \subseteq \kappa$ such that $S \cap \alpha$ is stationary in α for all $\alpha \in C$ with $cf(\alpha) > \omega$.

(a) Prove that $<$ is transitive.

(b) Prove that $<$ is well-founded.

(5) Carefuly state the Condensation Lemma, and use it to prove that GCH holds in L.

(6) Let U be a normal measure on the measurable class κ. Carefully define the tramsitive class $Ult(V, U)$, and explain why there is a non-trivial elementary embedding from V to $Ult(V, U)$.

Prove that if κ is measurable, U is a normal measure on κ and $M = Ult(V, U)$ then M is not closed under κ^+-sequences and $V_{\kappa+2} \not\subseteq M$.